F.un seminar

Fun with F.un (CCM) Talk by Bram Mesland 24/10/2008 ①

The rational BC-alg $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

given by $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ and generators

μ_n, μ_n^* with relations
$$\left.\begin{array}{l} \mu_n^* \mu_n = 1 \\ \mu_{nm} = \mu_n \mu_m \\ \mu_{nm}^* = \mu_n^* \mu_m^* \\ \mu_n \mu_m^* = \mu_m^* \mu_n \end{array}\right\}$$

$$\mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta=\gamma} e(\delta)$$

Consider the maps
$$\rho_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \longrightarrow \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$$
$$e(\gamma) \longmapsto \frac{1}{n} \sum_{n\delta=\gamma} e(\delta)$$

and the idempotent $\pi_n = \frac{1}{n} \sum_{s\in\Gamma_n} e(s)$ $\quad \Gamma_n \subseteq \mathbb{Q}/\mathbb{Z}$ n-torsion (n-th roots of 1)

$\pi_n^2 = \pi_n$, and $\rho_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \xrightarrow{\sim} \pi_n \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ isomorphism

Define $\sigma_n(e(\gamma)) := e(n\gamma)$

$\sigma_n \rho_n = \mathrm{Id}$
$\rho_n \sigma_n = \pi_n$ (actually, "multiplication by π_n")

σ_n's restrict to maps $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$

Let $\hat{\rho}_n := n\rho_n$, then if we replace μ_n by $\tilde{\mu}_n = n\mu_n$,

we get a new set of relations:

$\tilde{\mu}_n x \mu_n^* = \hat{\rho}_n(x)$	$\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m$	(Realize that we lost unitarity!)
$\mu_n^* x = \sigma_n(x)\mu_n^*$	$\mu_{nm}^* = \mu_n^* \mu_m^*$	$x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$
$x\tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x)$	$\mu_n^* \tilde{\mu}_n = n$	
	$\tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n$	

Define $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ by the generators and relations as above.

This is related to the tower of field extensions

$$\mathbb{F}_{1^n} \subseteq \mathbb{F}_{1^m} \subseteq \cdots \subseteq \mathbb{F}_{1^\infty} \qquad n \mid m, \text{ with endomorphisms}$$

coming from Frobenius-type maps.

Let $\mathcal{R}$ the full-subcat of the cat. of f.g. flat rings $/\mathbb{Z}$ generated by the rings $A_n = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ and their tensor products (a la Soulé)

Recall: A gadget $(\underline{X}, A, e)$ is given by ...

- $\underline{X} : \mathcal{R} \longrightarrow \text{Sets}$
- A complex alg.
- $e : \underline{X} \Rightarrow (R \to \mathrm{Hom}(A, R \otimes_{\mathbb{Z}} \mathbb{C}))$ natural transf.

The varieties $\mu^{(k)}$

$$\underline{\mu}^{(k)}(R) := \{ x \in R \mid x^k = 1 \} \qquad (k\text{-th roots of } 1 \text{ in } R)$$

$$\mu^{(k)} = F(\mathrm{Spec}(A_k)) \qquad (F : \{\text{Varieties}/\mathbb{Z}\} \longrightarrow \{\text{Schemes}/\mathbb{F}_1\})$$

this gives us an inductive system of varieties $/\mathbb{F}_1$

Define $\underline{\mu}^{(\infty)}(R) := \varprojlim \mathrm{Hom}(A_k, R) = \mathrm{Hom}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$

The maps $\sigma_n : A_k \longrightarrow A_k$

$T \longmapsto T^n$

are morphisms of the inductive system in the limit

$\sigma_n : A = \varinjlim A_k \longrightarrow A$ and here they are surjective.

<u>Prop</u>: The σ_n are morphisms of $\mathbb{F}_1$-varieties

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If $\mathbb{K}$ perfect field, $p = \mathrm{char}\, \mathbb{K} > 0$, then the map

$\sigma_{p^\ell} \otimes (\sigma_{\mathbb{F}_p})^\ell : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{K} \longrightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{K}$

~~acts by~~ acts as $f \longmapsto f^{p^\ell}$