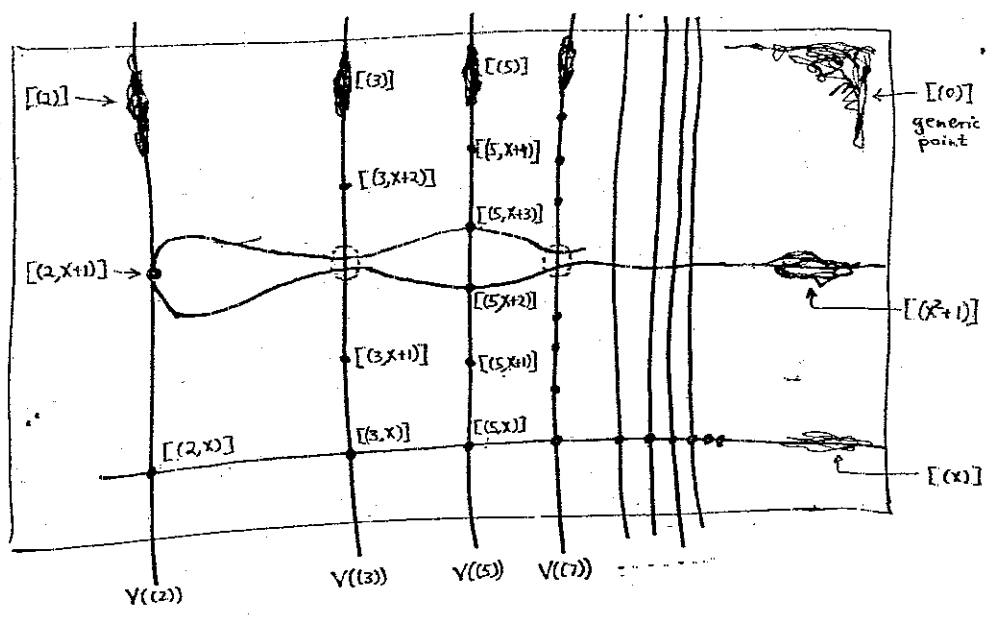


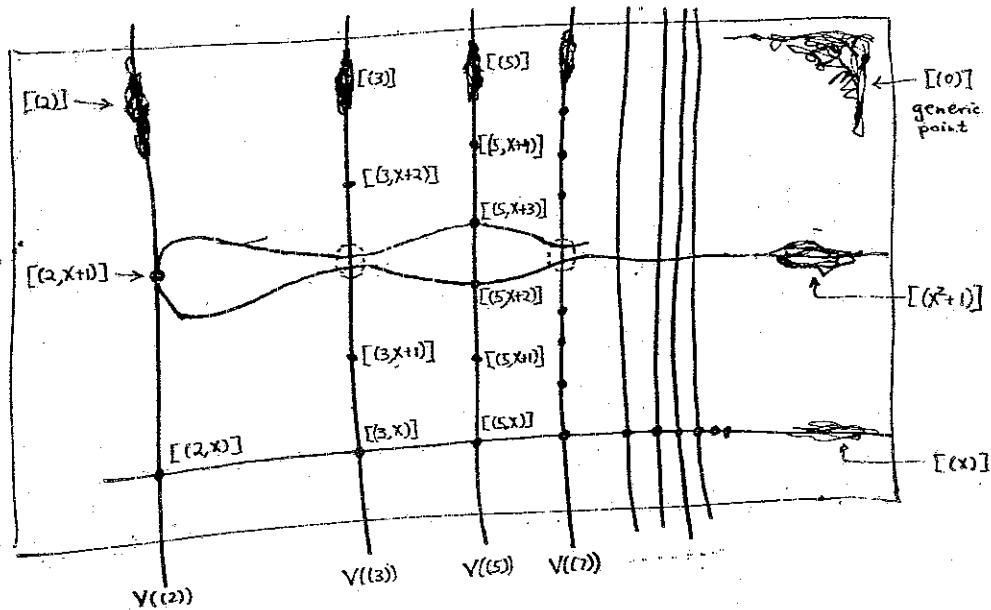
UN DESSIN D'ENFANCE



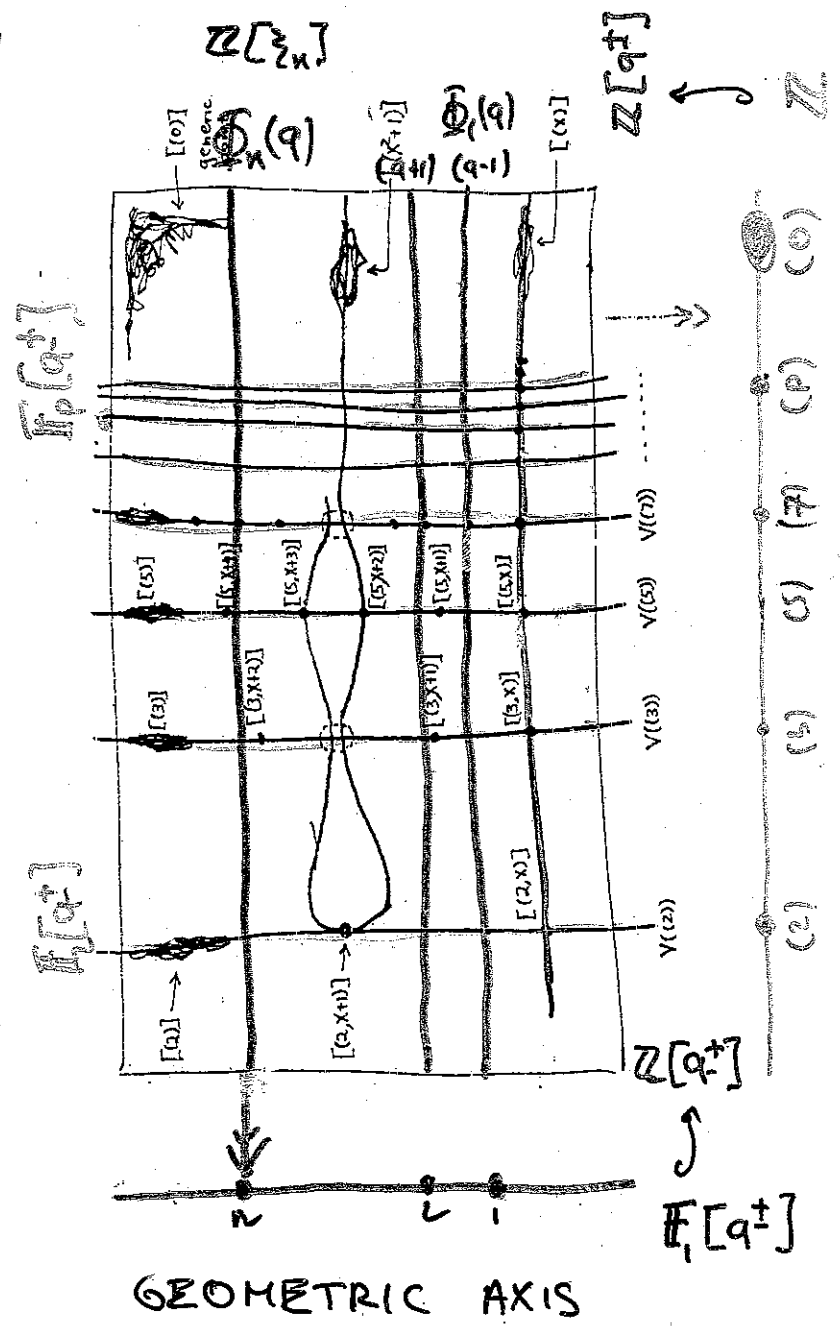
Example H: $\text{Spec}(\mathbb{Z}[X])$. This is a so-called "arithmetic surface" and is the first example which has a real mixing of arithmetic and geometric properties. The prime ideals in $\mathbb{Z}[X]$ are:

- i) (0) ,
- ii) principal prime ideals (f) , where f is either a prime p , or a \mathbb{Q} -irreducible polynomial written so that its coefficients have g.c.d. 1,
- iii) maximal ideals (p, f) , p a prime and f a monic integral polynomial irreducible modulo p .

The whole should be pictured as follows:



acyclotomic itygen
cyclotomic polys



Manin 0809.15.64 "cyclotomy & analytic geometry over \mathbb{F}_q "

BASIC ANALOGIES

Field k

Field \mathbb{F}_q

① vector space

set

② dimension

cardinality

③ $GL_n(k)$

S_n

④ G -representation

$G \rightarrow S_n$

$G \rightarrow GL_n(k)$

permutation repr.

⑤ irreducible
 G -representation

transitive
permutation repr.

⑥ k -field

cyclic group

⑦ k -algebra

abelian group

(noncommutative)

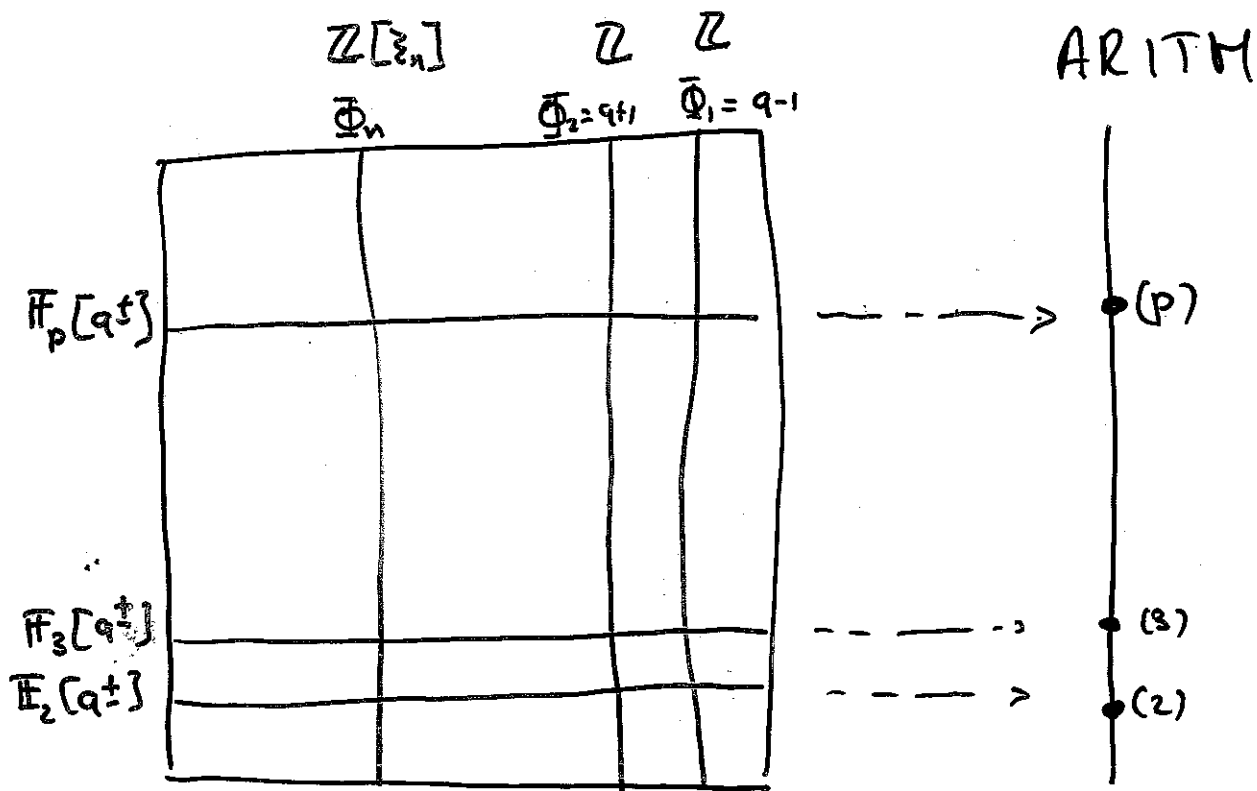
(group)

MANTRA

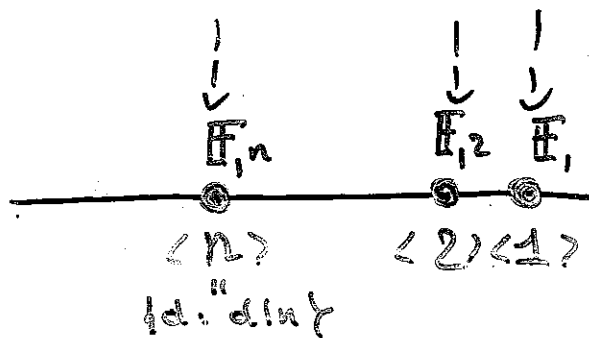
\mathbb{F}_1 -stuff acquires flesh after extending to $\mathbb{Z}(\alpha\mathbb{C})$

SOULÉ - PROPOSAL

$$\mathbb{F}_{\pm n} \otimes_{\mathbb{F}_1} \mathbb{Z} \cong \mathbb{Z} C_n$$



GEOM



SPEC $\mathbb{F}_1[q^\pm]$

↑
WHAT?

$$\mathbb{F}_{1,n} = \mathbb{F}_1[q^\pm] / (q^n - 1)$$

$$\text{fiber} = (q^n - 1) = \prod_{d|n} \Phi_d(q)$$

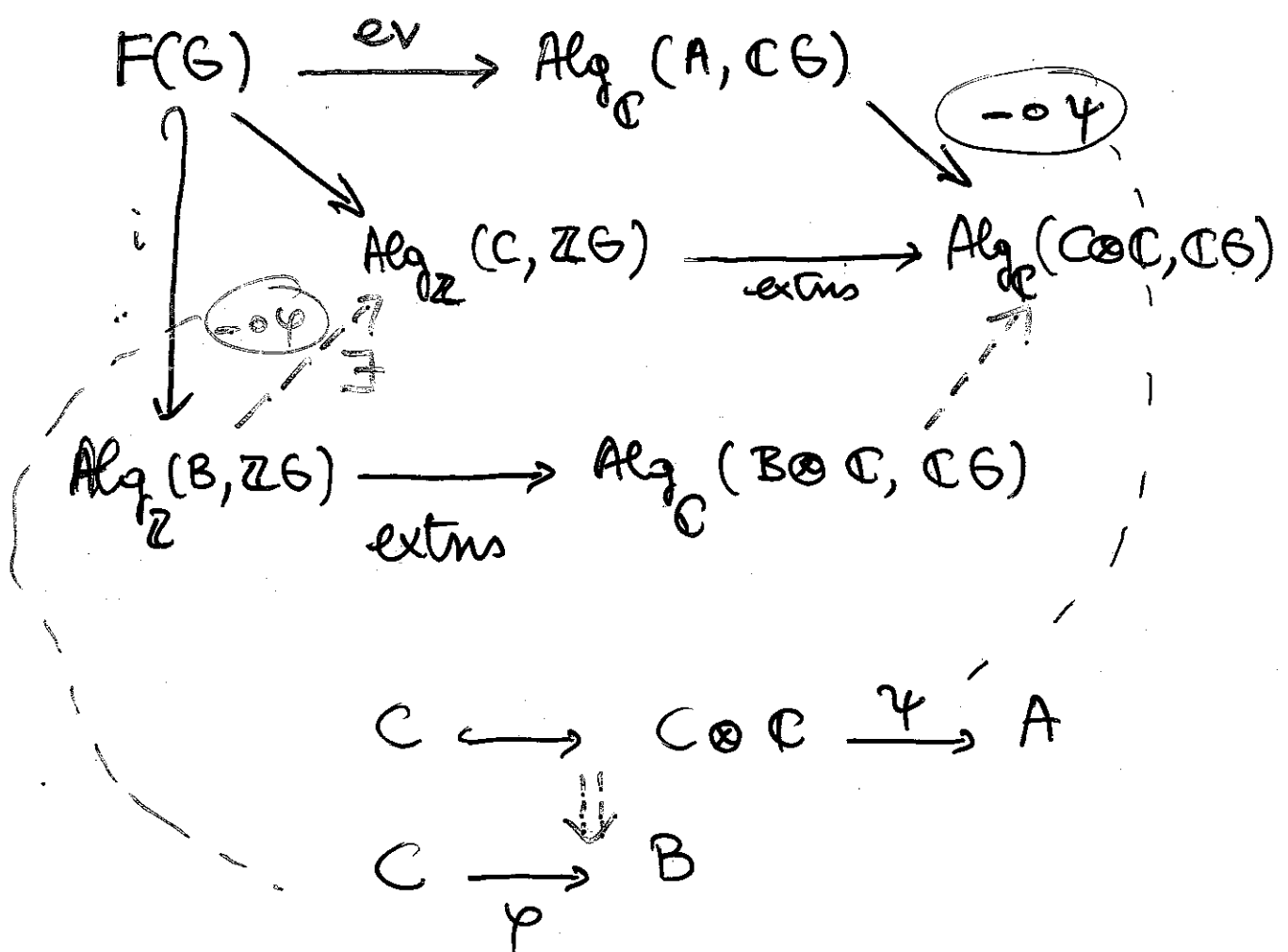
CONNES - CONSANI (arXiv: 0809.2926)

"On the notion of geometry over \mathbb{F} ,"

① $F : (\text{abelian}) \rightarrow (\text{sets})$ "functor of points"

② $\exists \mathbb{C}$ -algebra A s.t. $\forall G$
 $F(G) \xrightarrow{ev} \text{Alg}_{\mathbb{C}}(A, \mathbb{C}G)$ "realization w.r.t. archimedean valuation"

③ $\exists!$ \mathbb{Z} -algebra B s.t. $\forall G$ "Z-model"
 "best" $F(G) \xrightarrow{i} \text{Alg}_{\mathbb{Z}}(B, \mathbb{Z}G)$



[noncommutative \mathbb{F}_1 -geometry : groups]

$$\mathbb{F}_1[q^\pm] : \begin{cases} F = \text{forgetfull} \\ A = \mathbb{C}[q^\pm] \\ B = \mathbb{Z}[q^\pm] \end{cases}$$

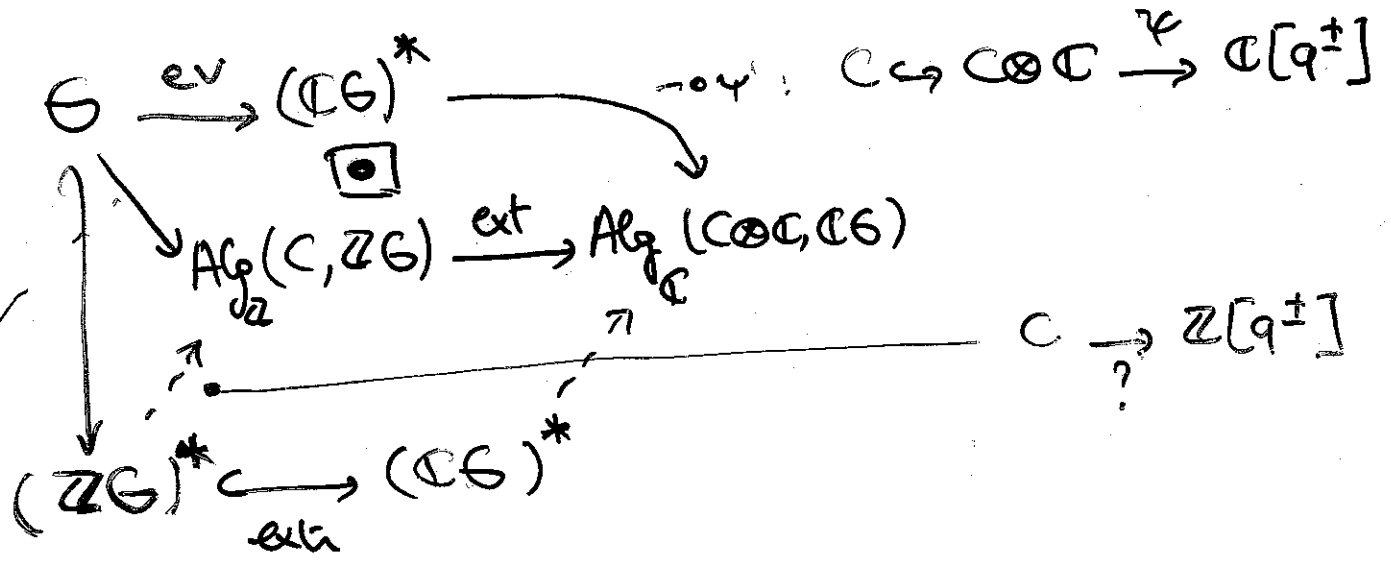
$$F(G) = G$$

$$G \xrightarrow{ev} (\mathbb{C}G)^*$$

$$g \mapsto e_g$$

$$G \xrightarrow{c} (\mathbb{Z}G)^*$$

$$g \mapsto e_g$$



TRICK: take $C_N = \langle c \rangle \quad N \gg$
and trace c around.

$$C \hookrightarrow C \otimes C \xrightarrow{\gamma} \mathbb{C}[q^\pm]$$

$$\downarrow \qquad \qquad \qquad \downarrow \pi$$

$$\mathbb{Z}C_N \longleftarrow \mathbb{C}C_N = \frac{\mathbb{C}[q^\pm]}{(q^N - 1)}$$

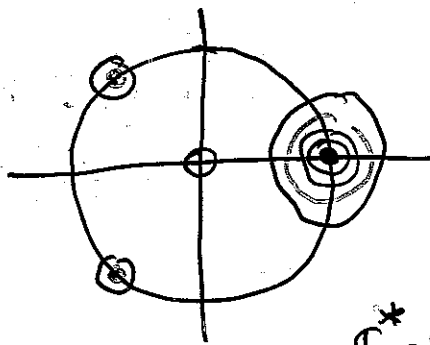
$\mathbb{F}_1[q^\pm]$ exists! (toric varieties, Chevalley groups, proj space...)

[dessins: $PSL_2(\mathbb{Z}) = C_2 * C_3 = \Gamma$; $\mathbb{C}\Gamma = A$, $B = \mathbb{Z}\Gamma$
 $F(\text{genus } m) \rightarrow (\text{sets}) \quad G \rightarrow G_2 \times G_3$]

COMPLEX

\mathbb{C}_3	e	e	e^2
χ_1	1	1	1
χ_2	1	ρ	ρ^2
χ_3	1	ρ^2	ρ

$\text{Spec } \mathbb{C}_3$

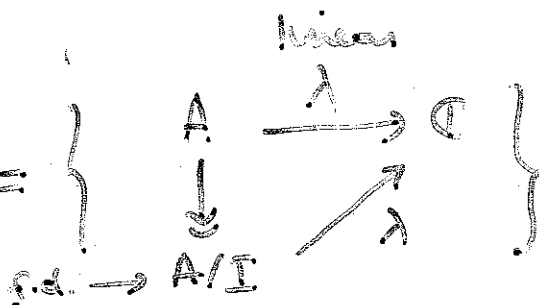


$\mathbb{C}^* = \text{Spec } \mathbb{C}[q^\pm]$

\xrightarrow{ev}

(fin. dim \mathbb{C} -alg) \rightarrow (sets)

Thin spectrum
 $\text{Spec}_{\text{Thin}}(A) = A^0 =$



$$\left(\frac{\mathbb{C}[q^\pm]}{((q-\alpha_1)^{n_1} \dots (q-\alpha_n)^{n_n})} \right)^* \cong \left(\frac{\mathbb{C}[q^\pm]}{(q-\alpha_1)^{n_1}} \right)^* \oplus \dots \oplus \left(\frac{\mathbb{C}[q^\pm]}{(q-\alpha_n)^{n_n}} \right)^*$$

$\mathbb{C}[q^\pm]^0 = \left[\bigoplus_{\alpha \in \mathbb{C}^*} \right] \cup (T_{\mathbb{C}^*, \alpha})$ \leftarrow tg w/o

Space = covered

$\mathbb{C}[q^\pm] \xleftrightarrow{\text{TAYLOR}} \mathbb{C}[q^\pm]^{0*} = \prod_{\alpha \in \mathbb{C}^*} \mathbb{C}[[q-\alpha]]$

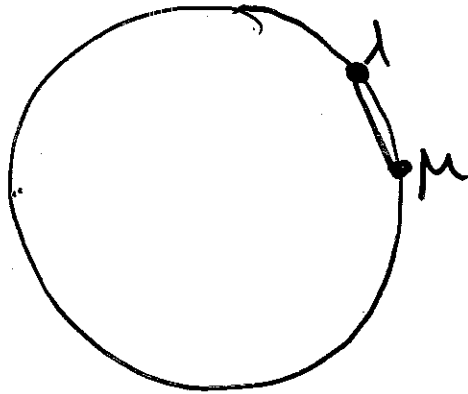
\mathbb{F}_1 -geometry \Rightarrow Complex picture

① COORDINATE RING

$$\mathbb{F}_1[q^\pm] \otimes_{\mathbb{F}_1} \mathbb{C} = \bigcap_{\alpha \in \mathbb{M}_\infty} \mathbb{C}[[q-\alpha]]$$

"holomorphic fctns defined in all roots of 1"

② NEW TOPOLOGY ON \mathbb{M}_∞



λ and μ "nearby"

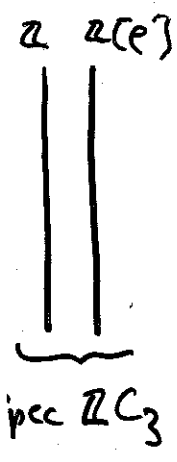
$$\lambda \leftrightarrow \mu$$

iff

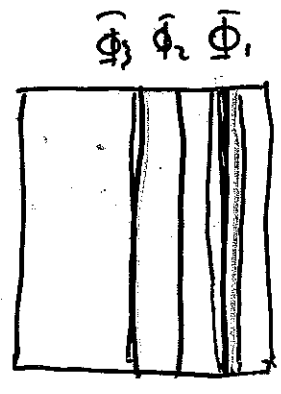
$\frac{\lambda}{\mu}$ of order p^k , $k \in \mathbb{Z}$

"nearness preserved under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ "

(usual topology: not!)



INTEGRAL



(f.g. proj \mathbb{Z} -rings) \rightarrow (sets)

!!

$\text{Spec}_{\text{THIN}}(B) = B^c = \left\{ \begin{array}{ccc} B & \xrightarrow{\lambda} & \mathbb{Z} \\ \downarrow & \nearrow \lambda & \\ B/I & \leftarrow & \text{f.g. proj } \mathbb{Z}\text{-ring} \end{array} \right\}$

$$\left(\frac{\mathbb{Z}[q^+]}{(p_1(q)^{n_1} \dots p_k(q)^{n_k})} \right)^* \iff \left(\frac{\mathbb{Z}[q^+]}{(p_1(q)^{n_1}} \right)^* \oplus \dots \oplus \left(\frac{\mathbb{Z}[q^+]}{(p_k(q)^{n_k}} \right)^*$$

$\hat{=}$ not necessarily co-maximal!

\mathbb{F}_1 -picture involves cyclotomic irred. $\Phi_n(q)$

$$\frac{m}{n} \neq p^k, k \in \mathbb{Z} \implies (\Phi_n(q), \Phi_m(q)) = 1$$

$$\frac{m}{n} = p^k, k \in \mathbb{N} \implies \Phi_m(q) \equiv \Phi_n(q)^d \pmod{p}$$

(exist extension $\mathbb{Z}[\xi_m] \supset \mathbb{Z}[\xi_n]$)

"clique" - relation on $\mathbb{N} = \{1, 2, 3, \dots\}$ induced

$$n \leftrightarrow m \text{ iff } \frac{m}{n} = p^k, k \in \mathbb{Z}$$

\mathbb{F}_1 -part of $\mathbb{Z}[q^\pm]^{0*}$ involves completions

$$\widehat{\mathbb{Z}[q^\pm]^S} = \varprojlim_{p \in \Phi_S^*} \mathbb{Z}[q^\pm]/(p)$$

$$S \subset \mathbb{N} \quad \Phi_S^* = \{ \text{monics gener. by } \Phi_n(q), n \in S \}$$

K. HABIRO 0209324 "cyclotomic completions of polynomial up"

① $S' \subset S$ s.t. every component of clique(S) has $\ell \in S'$

$$\Rightarrow \rho_{S'}^S : \widehat{\mathbb{Z}[q^\pm]^S} \longleftrightarrow \widehat{\mathbb{Z}[q^\pm]^{S'}}$$

② if for every $n \in S : \langle n \rangle = \{ m | n \mid m \} \subset S$, then

$$\widehat{\mathbb{Z}[q^\pm]^S} = \bigcap_{n \in S} \widehat{\mathbb{Z}[q^\pm]^{\langle n \rangle}} = \bigcap_{n \in S} \widehat{\mathbb{Z}[q^\pm]_{(q^n-1)}}$$

\mathbb{F}_1 -geometry \Rightarrow integral picture

$$\begin{aligned} \mathbb{F}_1[q^\pm] \otimes_{\mathbb{F}_1}^{\text{an}} \mathbb{Z} &= \bigcap_{n \in \mathbb{N}} \widehat{\mathbb{Z}[q^\pm]_{(q^n-1)}} \\ &= \widehat{\mathbb{Z}[q^\pm]^{\mathbb{N}}} \end{aligned}$$

$\mathbb{C} : \bigcap_{\alpha \in \mathbb{M}_{\infty}} \mathbb{C}[[q^{-\alpha}]]$ holom. fctns def. on \mathbb{M}_{∞}

$$\mathbb{Z} : \widehat{\mathbb{Z}[q^{\pm}]}^{\mathbb{N}} = \lim_{\leftarrow \mathbb{N}} \mathbb{Z}[q^{\pm}] / ((q^N - 1)(q^{N-1} - 1) \dots (q - 1))$$

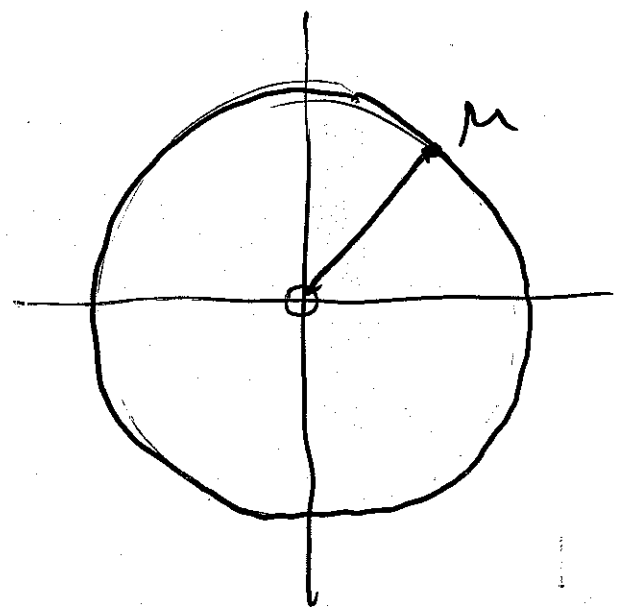
$$= \left\{ \sum_{N=0}^{\infty} a_N(q)(q^N - 1)(q^{N-1} - 1) \dots (q - 1) ; \deg a_N < N \right\} \subset \mathbb{Z}[[q^{\pm}]]$$

WELL DEFINED ON \mathbb{M}_{∞}

GAL($\overline{\mathbb{Q}}/\mathbb{Q}$)

Example : Kontsevich - Zagier

$$\sum_{n=0}^{\infty} (1-q)(1-q^2) \dots (1-q^n) = -\frac{1}{2} \sum_{n=0}^{\infty} n \pi(n) q^{\frac{n^2-1}{24}}$$



"DESSINS D'ENFANT"

$$\mathbb{Z} = \langle c \rangle = C_{\infty}$$

$$C_n$$

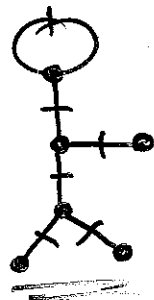
$$C_3 = \{e, c, c^2\}$$

$$\Gamma = \text{PSL}_2(\mathbb{Z}) = C_2 * C_3$$

dessins

$$X \xrightarrow{\mathbb{Z}} \mathbb{P}^1$$

M_{12}



(ab) \xrightarrow{F} (nets)
forget

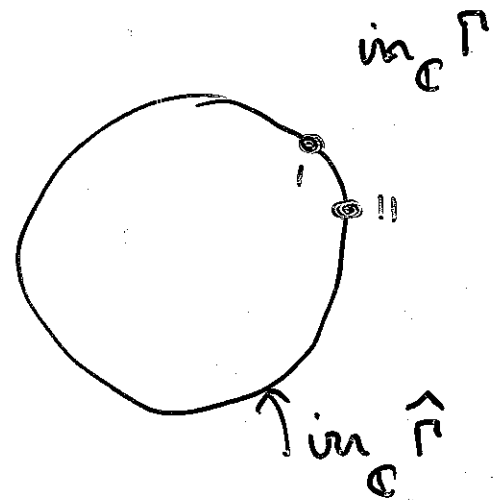
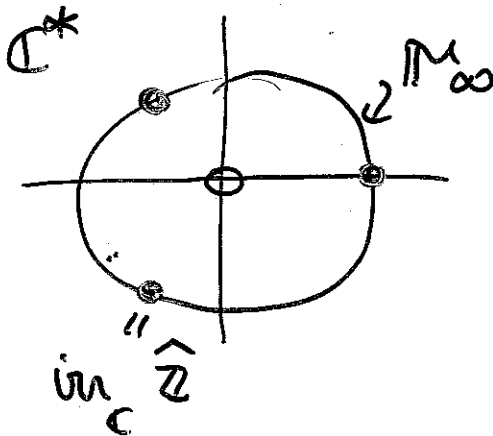
(groups) \rightarrow (nets)
 $G \rightarrow G_2 \times G_3$

$$\mathbb{C}[q^{\pm}]$$

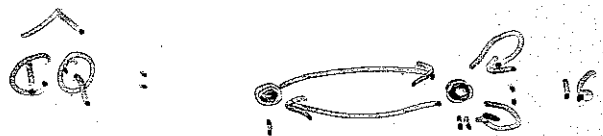
$$\mathbb{C}\Gamma$$

$$\mathbb{Z}[q^{\pm}]$$

$$\mathbb{Z}\Gamma$$



$$\mathbb{C}[q^{-1}] \oplus \mathbb{C}[q^{-p}] \oplus \mathbb{C}[q^{-p^2}]$$



$$\mathbb{C}[q^{\pm}]^{\circ}$$

path coalg of ext-quiver
subcoalg on $\text{in}_C \hat{\Gamma}$

$$\mathbb{F}_1[q^{\pm}]^{\text{an}} \otimes \mathbb{C}$$

(3)

$$\mathbb{Z}[q^{\pm}]^{\wedge \mathbb{N}}$$

???