Chevalley schemes are defined over $\mathbb{F}_{1^2}$

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These are rough notes of a talk I gave in Max-Planck Institute for Mathematics during the $\mathbb{F}_1$ study seminar. This is based on Connes and Consani's preprint ("On the notion of geometry over $\mathbb{F}_1$"), although I have freely interpreted some of the arguments.

1 Varieties over $\mathbb{F}_{1^n}$

"variety" over S : integral scheme over S.
If X is a scheme and R a ring, $X(R) = \mathrm{Mor}(\mathrm{Spec}\ R, X)$

Definition 1.1 Let us consider :

- $\mathcal{F}^n_{ab}$ is the category of pairs (D, ϵ) where D is a finite abelian group and $\epsilon \in D$ is such that $\epsilon^n = 1$.
- $\mathbb{C}[D, \epsilon] = \mathbb{C}[D] \otimes_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]} \mathbb{C} = \mathbb{C}[D]/(e_\epsilon - e^{2i\pi/n})$
- If $X_\mathbb{C}$ is a complex variety then $\underline{X_\mathbb{C}}$ is the covariant functor $\mathcal{F}^n_{ab} \to (\mathrm{Sets})$ defined by $\underline{X_\mathbb{C}}(D, \epsilon) = \mathrm{Mor}(\mathrm{Spec}\ \mathbb{C}[D, \epsilon], X_\mathbb{C})$

Definition 1.2 A gadget over $\mathbb{F}_{1^n}$ is a triple $X = (\underline{X}, X_\mathbb{C}, e_X)$ where

- $\underline{X}$ is a covariant functor $\mathcal{F}^n_{ab} \to (\mathrm{Sets})$.
- $X_\mathbb{C}$ is a complex variety.
- e_X is a natural transformation $\underline{X} \to \underline{X_\mathbb{C}}$.

Note than when $n = 1$ one recovers the definition of gadgets over $\mathbb{F}_1$.

Definition 1.3 X is finite if $\underline{X}(D, \epsilon)$ si finite for all (D, ϵ).

Definition 1.4 A morphism of gadgets $\Phi : X \to Y$ is a pair $(\underline{\Phi}, \Phi_\mathbb{C})$ where

- $\underline{\Phi} : \underline{X} \to \underline{Y}$ is a natural transformation.
- $\Phi_\mathbb{C} : X_\mathbb{C} \to Y_\mathbb{C}$ is a morphism of varieties.
- $\forall (D, \epsilon)$ the following diagram commutes :

$$\begin{array}{ccc} \underline{X}(D,\epsilon) & \xrightarrow{\Phi(D)} & \underline{Y}(D,\epsilon) \\ \downarrow{\scriptstyle e_X(D,\epsilon)} & & \downarrow{\scriptstyle e_Y(D,\epsilon)} \\ X_\mathbb{C}(\mathbb{C}[D,\epsilon]) & \xrightarrow{\Phi_\mathbb{C}} & Y_\mathbb{C}(\mathbb{C}[D,\epsilon]) \end{array}$$

For example, an affine variety over $\mathbb{Z}$ defines a gadget. Namely, we have a functor $\mathcal{G}$: (Var/$\mathbb{Z}$) $\to$ (Gadgets over $\mathbb{F}_{1^n}$) which maps a variety V to the triple

$$\mathcal{G}(V) := (Hom(\mathbb{Z}[V], \mathbb{C}[D, \epsilon]), V_{\mathbb{C}}, - \otimes_{\mathbb{Z}} \mathbb{C})$$

Note however that in general $\mathcal{G}(V)$ is not finite since $\mathcal{G}(\mathbb{A}^1_{\mathbb{Z}})(\{1\}) = \text{Mor}(\text{Spec } \mathbb{Z}, \mathbb{A}^1_{\mathbb{Z}}) = \mathbb{Z}$.

We have a base change functor from (Gadgets over $\mathbb{F}_{1^n}$) to (Gadgets over $\mathbb{F}_{1^{nm}}$) whenever n, m are positive integers.

Definition 1.5 Φ is an immersion if

- $\underline{\Phi}(D, \epsilon)$ is injective.
- $\Phi_{\mathbb{C}}$ is an immersion.

Definition 1.6 A gadget $X = (\underline{X}, X_{\mathbb{C}}, e_X)$ over $\mathbb{F}_{1^n}$ is an affine variety over $\mathbb{F}_{1^n}$ if it is finite and there exists an affine variety $X_{\mathbb{Z}}$ over $\mathbb{Z}$ and an open immersion of gadgets $i : X \to \mathcal{G}(X_{\mathbb{Z}})$ such that for all $V_{\mathbb{Z}}$ and $\varphi : X \to \mathcal{G}(V_{\mathbb{Z}})$ there exists a unique $\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \to V_{\mathbb{Z}}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \mathcal{G}(X_{\mathbb{Z}}) \\ {\scriptstyle \forall\psi}\downarrow & \swarrow {\scriptstyle \exists!\ \mathcal{G}(\varphi_{\mathbb{Z}})} & \\ \mathcal{G}(V_{\mathbb{Z}}) & & \end{array}$$

Proposition 1.7 *The multiplicative group $\mathbb{G}_m$ and the additive group $\mathbb{G}_a$ may be defined as affine varieties over $\mathbb{F}_1$ by setting*

$\underline{\mathbb{G}_m}(D) = D$	$\underline{\mathbb{G}_a}(D) = D \cup \{0\}$
$\mathbb{G}_{m,\mathbb{C}} = \mathit{Spec}\ \mathbb{C}[T, T^{-1}]$	$\mathbb{G}_{a,\mathbb{C}} = \mathit{Spec}\ \mathbb{C}[T]$
$\forall d \in D, e_m(d) : \mathbb{C}[T, T^{-1}] \ni T \mapsto d \in \mathbb{C}[D]$	$\begin{cases} \forall d \in D, e_m(d) : \mathbb{C}[T] \ni T \mapsto d \in \mathbb{C}[D] \\ e_m(0) : \mathbb{C}[T] \ni T \mapsto 0 \in \mathbb{C}[D] \end{cases}$

Note however that the group structure $G \times G \to G$ is a morphism of varieties over $\mathbb{F}_1$ for $\mathbb{G}_m$ but NOT for $\mathbb{G}_a$!! In fact, if $X = (\underline{X}, X_{\mathbb{C}}, e_X)$ is a gadget over $\mathbb{F}_1$, we denote $X(\mathbb{F}_1) := e_X(\underline{X}(\{1\})$: this is a well-defined subset of $X_{\mathbb{C}}(\mathbb{C})$ and if $\Phi = (\underline{\Phi}, \Phi_{\mathbb{C}}) : X \to Y$ is a morphism of gadget, then $\Phi_{\mathbb{C}}(X(\mathbb{F}_1)) \subset Y(\mathbb{F}_1)$. We have $\mathbb{G}_a(\mathbb{F}_1) = \{0, 1\}$ and so the multiplication $\mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ is not a morphism of gadgets.

Proof. Let us prove that the gadget $\mathbb{G}_m$ is a variety over $\mathbb{F}_1$. We have an immersion from the gadget $\mathbb{G}_m$ to the gadget $\mathcal{G}(\mathbb{G}_{m,\mathbb{Z}})$. Let us show that it satisfies the universal property of affine varieties over $\mathbb{F}_1$.

Let $V_{\mathbb{Z}}$ be an affine variety over $\mathbb{Z}$ and let $\psi : \mathbb{G}_m \to \mathcal{G}(V_{\mathbb{Z}})$ be a morphism of gadgets. We have to show the existence and unicity a morphism $\varphi_{\mathbb{Z}} : \mathbb{G}_{m,\mathbb{Z}} \to V$ such that (1.6) holds. The unicity is clear since $\mathbb{G}_{m,\mathbb{C}}$ is dense in $\mathbb{G}_{m,\mathbb{Z}}$ and the complexification of $\varphi_{\mathbb{Z}}$ must be $\psi_{\mathbb{C}}$. The existence of $\varphi_{\mathbb{Z}}$ amounts to the fact that $\psi_{\mathbb{C}}$ is defined over $\mathbb{Z}$, or that $\psi^{\#}_{\mathbb{C}}(\mathcal{O}_{\mathbb{Z}}[V]) \subset \mathcal{O}_{\mathbb{Z}}[\mathbb{G}_m]$.

The fact that ψ is a morphism of gadgets means that

$$\underline{\psi}(D)(\bar{1}) = e_m(D)(\bar{1}) \circ \psi^{\#}_{\mathbb{C}} \in Hom_{\mathbb{C}-alg}(\mathcal{O}_{\mathbb{C}}[V], \mathbb{C}[X]/(X^n - 1)).$$

Let $h \in \mathcal{O}_{\mathbb{Z}}[V]$. Since $e_m(D)(\bar{1}) \in \mathcal{G}(V_{\mathbb{Z}})(D)$, we have $\underline{\psi}(D)(\bar{1})(h) \in \mathbb{Z}[X^n]/(X^n - 1)$. On the other hand, $e_m(\bar{1})$ maps $T \in \mathbb{C}[T, T^{-1}]$ on $X \in \mathbb{C}[X]/(X^n - 1)$. Thus the image of $\psi^{\#}_{\mathbb{C}}(h) \in \mathbb{C}[T, T^{-1}]$ in $\mathbb{C}[X]/(X^n - 1)$ lies in $\mathbb{Z}[X]/(X^n - 1)$. Since this holds for all n, $\psi^{\#}_{\mathbb{C}}(h)$ itself lies in $\mathbb{Z}[T, T^{-1}]$ and this proves the proposition. □

Proposition 1.8 *The product of varieties over $\mathbb{F}_{1^n}$ is a variety over $\mathbb{F}_{1^n}$.*

2 $N_G(T)$ as an affine variety over $\mathbb{F}_{1^2}$

$G_{\mathbb{Z}}$ (simply-connected) Chevalley group scheme over $\mathbb{Z}$ (oral : group scheme over $\mathbb{Z}$, smooth, reductive, split)
$T \subset G_{\mathbb{Z}}$ a split maximal torus, $N = N_{G_{\mathbb{Z}}}(T)$ its normaliser, and set $p : N \to W = N/T$.
example of SL_n
Our goal is to define N over $\mathbb{F}_{1^2}$. Remark that intuitively we don't expect that G can be defined over $\mathbb{F}_1$, since then we would have, setting $N(\mathbb{F}_1) := e_N(\underline{N}(\{1\})$, $N(\mathbb{F}_1) \subset G(\mathbb{F}_1) \subset G_{\mathbb{C}}$, and since we expect $G(\mathbb{F}_1) = W$, this would yield a canonical section of $N \to W$.

Definition 2.1 We define the functor $\underline{N}$ as follows :

- $\underline{N}(\{\pm 1\}, -1) := N(\mathbb{Z})$
- Let L be the character group of T, such that $T = \text{Spec } \mathbb{Z}[L]$.
- Let $\varphi_{\{\pm 1\} \subset \mathbb{Z}} : Hom(L, \{\pm 1\}) \to T(\mathbb{Z})$ be defined as the composition of the following maps :

$$Hom_{\mathbb{Z}}(L, \{\pm 1\}) \to Hom_{\mathbb{Z}-alg}(\mathbb{Z}[L], \mathbb{Z}) = \text{Mor}(\text{Spec } \mathbb{Z}, T) = T(\mathbb{Z}).$$

- Let $N(D, \epsilon) = (Hom(L, D) \times N(\mathbb{Z}))/Hom(L, \{\pm 1\})$, where $Hom(L, \{\pm 1\}) \to Hom(L, D)$ is induced by $\{\pm 1\} \to D, -1 \mapsto \epsilon$ and $\varphi_{\{\pm 1\} \subset \mathbb{Z}}$ yields $Hom(L, \{\pm 1\}) \to T(\mathbb{Z}) \subset N(\mathbb{Z})$.
- For $u \in Hom(L, D)$ define $e_T(D, \epsilon)(u) \in \text{Mor}(\text{Spec } \mathbb{C}[D, \epsilon], T)$ by the composition $\mathbb{C}[L] \to \mathbb{C}[D] \to \mathbb{C}[D, \epsilon]$, where $\mathbb{C}[L] \to \mathbb{C}[D]$ maps e_l on $e_{u(l)}$.
- The map $Hom(L, D) \times N(\mathbb{Z})) \to \text{Mor}(\text{Spec } \mathbb{C}[D, \epsilon], N), (u, n) \mapsto n.e_T(D, \epsilon)(u)$ is invariant under $Hom(L, \{\pm 1\})$ and thus defines our natural transformation $\underline{e}_N : \underline{N}(D, \epsilon) \to \text{Mor}(\text{Spec } \mathbb{C}[D, \epsilon], N)$.

Proof that it is $Hom(L, \{\pm 1\})$-invariant :
it is enough to consider the case when $n \in T(\mathbb{Z})$, thus to prove that if $t \in Hom(L, D)$ and $u \in Hom(L, \{\pm 1\})$, then $e_T(D, \epsilon)(ut) = \varphi_{\{\pm 1\} \subset \mathbb{C}}(u).e_N(D, \epsilon)(t)$.

The homomorphism $e_N(D, \epsilon)(ut) \in Hom(\mathbb{Z}[L], \mathbb{C}[D, \epsilon])$ is given by mapping e_l on

$$\begin{cases} e_{t(l)} & \text{if} \quad u(l) = 1 \\ e_{\epsilon.t(l)} & \text{if} \quad u(l) = -1 \end{cases}$$

On the other hand the homomorphism $\varphi_{\{\pm 1\} \subset \mathbb{C}}(u).e_T(D, \epsilon)(t)$ is given by mapping e_l on $u(l)e_{t(l)}$. Since in $\mathbb{C}[D, \epsilon]$ we have $e_\epsilon = -1$, both are indeed equal. □

3 Defining Chevalley schemes over $\mathbb{F}_{1^2}$

By the Bruhat decomposition theorem, we know that Chevalley schemes are covered by cells which are products of affine spaces and tori. So the idea is to take the union of the corresponding varieties over $\mathbb{F}_{1^2}$. This construction can be given a functorial meaning only for $\mathbb{F}_{1^2}$, and not $\mathbb{F}_1$.

Let $R = R_+ \cup R_-$ be the set of roots.

Definition 3.1 We define the following subgroups :

- For $\alpha \in R$, $U(\alpha) \subset G$ is the corresponding additive group.

- For $w \in W$, set $R_w = \{\alpha \in R_+ : w(\alpha) \in R_-\}$.
- Set $U(w) = \prod_{\alpha \in R_w} U(\alpha)$.
- Set $U = \prod_{\alpha \in R_+} U(\alpha)$.

Theorem 1 (Chevalley) *Let k be a field. Then $G(k)$ is the disjoint union of the cells $C_w = U(k)T(k)wU_w(k)$.*

Example of SL_n.

Definition 3.2 We define the gadget $G = (\underline{G}, G_{\mathbb{C}}, e_G)$ by

- $\underline{G}(D, \epsilon) = \mathbb{A}^{R_+}(D) \times \bigcup_{w \in W}(p^{-1}(w) \times \mathbb{A}^{R_w}(D)$.
- $G_{\mathbb{C}}$ is the complexification of the group scheme $G_{\mathbb{Z}}$.
- For a given pair (D, ϵ) in $\mathcal{F}^n_{ab}$, $n \in \underline{N}(D, \epsilon)$ with $p(n) = w$, $d_+ \in D^{R_+}$ and $d_w \in D^{R_w}$, there are associated morphisms

  $$e_N(D,\epsilon)(n)\ ,\ e_{\mathbb{A}^{R_+}}(D,\epsilon)(d_+)\ ,\ e_{\mathbb{A}^{R_w}}(D,\epsilon)(d_w) \quad : \quad \text{Spec } \mathbb{C}[D,\epsilon] \to G\ ;$$

  we denote $e_G(d_+, n, d_w)$ the morphism Spec $\mathbb{C}[D, \epsilon] \to G$ obtained by pointwise multiplication in G.

Theorem 2 (Connes-Consani) *The gadget $G = (\underline{G}, G_{\mathbb{C}}, e_G)$ defines a variety over $\mathbb{F}_{1^2}$.*

Proof. We have an immersion of G in $\mathcal{G}(G_{\mathbb{Z}})$, and let us prove that this immersion satisfies the universal property.

Let V be an affine scheme over $\mathbb{Z}$ and let $\psi : G \to \mathcal{G}(V)$ be a morphism of gadgets. We have to show that $\psi_{\mathbb{C}}$ is defined over $\mathbb{Z}$. Let $w'_0 \in N(\mathbb{Z})$ such that $p(w'_0)$ is the element of maximal length in W. Then ψ restricts on the open cell $G^0 := UTw'_0U$ to a morphism of gadgets $G^0 \simeq \mathbb{A}^{R_+} \times T \times \mathbb{A}^{R_+} \to \mathcal{G}(V)$. By the universal property for the affine variety $G^0 \simeq \mathbb{A}^{R_+} \times T \times \mathbb{A}^{R_+}$ over $\mathbb{F}_1$ (T is a product of multiplicative groups) this implies that $(\psi_{\mathbb{C}})^{\#}_{|G^0}$ maps $\mathbb{Z}[V]$ on $\mathbb{Z}[G^0]$. Recall now that G^0 is a principal open subset in G: there exists $\delta \in \mathbb{Z}[G]$ such that $\mathbb{Z}[G^0] \simeq \mathbb{Z}[G][\delta^{-1}]$. Since an element in $(\psi_{\mathbb{C}})^{\#}(\mathbb{Z}[V])$ cannot have poles, it follows that $(\psi_{\mathbb{C}})^{\#}(\mathbb{Z}[V]) \subset \mathbb{Z}[G]$, proving that $\psi_{\mathbb{C}}$ is in fact defined over $\mathbb{Z}$. □

Comment : we only used the open cell !!